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Continuous Dependence of the Solutions of an Ordinary Differential Equation

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1. INTRODUCTION

An interesting problem in the theory of ordinary differential equations involves characterizing a topology on a given set of right-hand sides such that the function taking each right-hand side into the solution (or set of solutions) of the associated differential equation is continuous when an appropriate notion of convergence is defined on its range.

Considering a given set D of right-hand sides to be made up of functions of the form

$$f: T \times \mathbb{R}^n \to \mathbb{R}^n$$

where $T = [t_1, t_2]$ is a fixed and compact interval in R, and each $f(\cdot, \cdot)$ is a Caratheodory function, the topologies which can be imposed on D so as to have the continuous dependence property indicated above must depend on additional conditions in the definition of D. For example, when D satisfies a condition which deals essentially with boundedness from above of $|f(t, x)|$ by a function which is integrable on T, as f varies on D and x varies on bounded subsets on $\mathbb{R}^n$, then a topology on D which is particularly useful for continuous dependence problems is a topology of joint continuity on compacts. This and similar topologies have been quite extensively studied in [1] and [2] where very general continuous dependence theorems have been obtained for Volterra integral equations. When further conditions are imposed on D, dealing mainly with the equicontinuity of the set of functions

$$x \to f(t, x)$$

as t varies on T and f varies on D, then the description of the topology which gives continuous dependence can be simplified. In fact, if $\mathscr{F}$ is the topology on D which is generated by subbasic sets of the form

$$\left\{ f \text{ in } D \,\middle|\, \int_{t_1}^{t} f(\tau, x)\, d\tau \text{ in } U \right\}$$

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for points x in $\mathbb{R}^n$, t in T, and open sets U in $\mathbb{R}^n$, then topology $\mathscr{E}$ gives continuous dependence in the following sense: if $\{f_\alpha\}$ is a net in D which converges with respect to $\mathscr{E}$ to a Caratheodory function f such that the differential equation

$$\dot{x}(t) = f(t, x(t)) \qquad \text{subject to } x(t_0) = x_0 \tag{1.1}$$

for some initial data point (t_0, x_0) in $T \times \mathbb{R}^n$ has a unique solution defined on T, then the net $\{x_\alpha(\cdot)\}$ converges to $x(\cdot)$ uniformly on T, where, for each α, $x_\alpha(\cdot)$ satisfies

$$\dot{x}(t) = f_\alpha(t, x(t)) \qquad \text{subject to } x(t_0) = x_0.$$

Moreover, this continuous dependence holds for all initial data points (t_0, x_0) in $T \times \mathbb{R}^n$ provided that the system (1.1) has a unique solution on T for each (t_0, x_0). Various forms of this result have been given in [3–6]. The topology $\mathscr{E}$ is particularly interesting because on suitable subsets D of right-hand sides it turns out to be the weakest among all topologies on D which give continuous dependence for each initial data point. Indeed, in [6] it is shown that this is the case when D is a set of right-hand sides of the form

$$\begin{aligned} \{f \mid\ & f(t,x) \text{ is measurable in } t \text{ and continuous in } x, \\ & |f(t,x)| \leqslant M(t) \\ & \text{and } |f(t,x) - f(t,x')| \leqslant K(t)\,|x - x'| \\ & \text{a.e. on } T, \text{ for all } x \text{ and } x' \text{ in } \mathbb{R}^n\}, \end{aligned}$$

where $M(\cdot)$ and $K(\cdot)$ are nonnegative integrable functions on T.

The object of the present note is to investigate other collections of right-hand sides on which a weakest topology for continuous dependence exists. It will be shown that when a set of right-hand sides is precompact with respect to a compact-open topology on the set of all functions satisfying the usual Caratheodory conditions, then there exists a weakest topology giving continuous dependence globally with respect to the initial data. Moreover, this weakest topology turns out to be $\mathscr{E}$, the topology indicated above, which is identical to the relativisation of the compact-open topology on the given set of right-hand sides.

This paper is organised as follows: Section 2 contains definitions and a statement of a preliminary result, Section 3 gives the main claim which asserts that a weakest topology of continuous dependence exists on sets of right-hand sides which are precompact in an appropriate compact-open topology on the set of all right-hand sides, Section 4 gives conditions which imply the compactness required for the results of Section 3, and Section 5 contains general remarks on another approach to the continuous dependence problem which has been reported in the literature.

2. Topologies and a Set of Right-Hand Sides. Definitions

Let T be a fixed compact interval $[t_1, t_2]$ of $\mathbb{R}$, Σ be the collection of Lebesgue measurable subsets of T, and let $\mu(\cdot)$ be the Lebesgue measure on T. Let X denote the Banach space of $\mathbb{R}^n$-valued continuous functions on T, and let $\|\cdot\|$ be the componentwise supremum norm on X. Points in $\mathbb{R}^n$ will be denoted by x, and points in X by $x(\cdot)$. However, when there is no danger of ambiguity notation will be abused and x will also be used to denote points in X.

Let C be the linear vector space of equivalence classes of functions $f\colon T \times \mathbb{R}^n \to \mathbb{R}^n$ which satisfy the following conditions:

1. for each x in $\mathbb{R}^n, f(\cdot, x)$ is measurable on T;
2. for each t in $T, f(t, \cdot)$ is continuous on $\mathbb{R}^n$;
3. for each $\eta > 0$ the set of functions

$$\{f(\cdot, x(\cdot)); \|x\| \leqslant \eta\}$$

is uniformly integrable on T, i.e., for each $\varepsilon > 0$ there exists some $\delta(\varepsilon) > 0$ such that if $\mu(E) < \delta(\varepsilon)$ then

$$\left| \int_E f(t, x(t))\, dt \right| < \varepsilon$$

for all x in X which satisfy $\|x\| \leqslant \eta$.

In view of conditions 2 and 3 and the Vitali convergence theorem, it follows that the function

$$x \to \int_E f(t, x(t))\, dt$$

is continuous on X for each E in Σ.

Two functions f and f' in C will be regarded as being in the same equivalence class if and only if

$$f(\cdot, x) = f'(\cdot, x)$$

a.e. on T for each x in $\mathbb{R}^n$. Henceforth, no distinction will be made between an equivalence class in C and a function f which belongs to it, i.e., the equivalence class will be denoted by f. For each initial data pair (t_0, x_0) and right-hand side f in C, the conditions 1 to 3 and the results of [7] imply the existence of a maximally defined solution of the differential equation

$$\dot{x}(t) = f(t, x(t)) \qquad \text{a.e. in the domain of } x(\cdot), \text{ subject to } x(t_0) = x_0.$$

By a maximally defined solution is meant a continuous function defined on a subinterval of T, which satisfies the initial condition and the differential equation on this subinterval, and which cannot be extended beyond the ends of the subinterval as a continuous function which still satisfies the equation.

Two linear topologies on C will be of interest. Let $\mathscr{T}$ be the topology on C generated by subbasic sets of the form

$$\left\{ f \text{ in } C; \int_E f(t, x)\, dt \text{ in } U \right\}$$

for each E in Σ where $\mu(E) > 0$, x in $\mathbb{R}^n$, and open sets U in $\mathbb{R}^n$. Let $\mathscr{T}_1$ be the compact-open topology on C generated by the subbase

$$\left\{ f \text{ in } C; \int_E f(t, x(t))\, dt \text{ in } U \text{ for all } x(\cdot) \text{ in } K \right\}$$

for each E in Σ where $\mu(E) > 0$, K is a compact subset in X, and U is an open set in $\mathbb{R}^n$. Clearly $\mathscr{T}$ is weaker than $\mathscr{T}_1$ and the definition of an equivalence class in C implies that they are Hausdorff. A reference for the notion of a topology generated by a subbase, weak versus strong topology, Hausdorff topology, and the concept of a net (to be used in what follows) is [8].

The topology $\mathscr{T}_1$ has a number of useful features. First, convergence of a net $\{f_\alpha\}$ to a limit f is equivalent to the convergence of the net of functions

$$x \to \int_E f_\alpha(t, x(t))\, dt$$

to the function

$$x \to \int_E f(t, x(t))\, dt$$

uniformly on compact subsets of X, for each E in Σ. This fact is easily verified directly (it may also be seen using [8, Theorem 7.11] and the fact that the weak topology on an L_1-space is generated by a uniformity). Second, $\mathscr{T}_1$ is jointly continuous on compact subsets of X. To see this, let $\{f_\alpha\}$ be a net in C which is $\mathscr{T}_1$-convergent to a limit f, and let $\{y_\beta\}$ be a net contained in a compact subset of X which converges to a limit y (in X). Then, from what was noted above, for each E in Σ

$$\lim_\alpha \int_E f_\alpha(t, y_\beta(t))\, dt = \int_E f(t, y_\beta(t))\, dt$$

uniformly with respect to β. Moreover,

$$\lim_{\beta} \int_E f_\alpha(t, y_\beta(t))\, dt = \int_E f_\alpha(t, y(t))\, dt$$

for each α. Thus it follows that

$$\lim_{\alpha \times \beta} \int_E f_\alpha(t, y_\beta(t))\, dt = \int_E f(t, y(t))\, dt$$

so that the function

$$(f, x) \to \int_E f(t, x(t))\, dt$$

is jointly continuous on compact subsets of X.

Let D be a given subset of C. Then a topology $\mathscr{U}$ on C is said to generate a topology of continuous dependence on D if, for each initial data point (t_0, x_0) in $T \times \mathbb{R}^n$ and each net $\{f_\alpha\}$ in D which is $\mathscr{U}$-convergent to a limit f in C, the following is true: if $\{f_\beta\}$ is an arbitrary subnet of $\{f_\alpha\}$ and $x(t_0, x_0; f_\beta)(\cdot)$ is a maximally defined solution of the differential equation

$$\dot{x}(t) = f_\beta(t, x(t)) \qquad \text{subject to } x(t_0) = x_0$$

for each β, then there exists a maximally defined solution $x(t_0, x_0 : f)(\cdot)$ of the differential equation

$$\dot{x}(t) = f(t, x(t)) \qquad \text{subject to } x(t_0) = x_0$$

and a subnet $\{f_\gamma\}$ of $\{f_\beta\}$ such that $\{x(t_0, x_0; f_\gamma)(\cdot)\}$ converges to $x(t_0, x_0; f)(\cdot)$ uniformly on compact subintervals of the maximal domain of definition of $x(t_0, x_0; if)(\cdot)$.

The following claim is a direct consequence of [2, Theorem 4.A] and the fact that $\mathscr{E}_1$ is jointly continuous on compact subsets of X:

Proposition 2.1. *Let D be a subset of C which satisfies the following condition: for each $\eta > 0$ the set of L_1-functions*

$$\{f(\cdot, x(\cdot)); f \text{ in } D, \|x\| < \eta\}$$

is uniformly integrable. Then the compact-open topology $\mathscr{E}_1$ generates a topology of continuous dependence on D.

3. Weakest Topology for Continuous Dependence

A subset $D \subset C$ is said to be $\mathscr{E}_1$-precompact in C if each net in D contains a subnet which is $\mathscr{E}_1$-convergent to an element of C. The object of the present section is to study the continuous dependence problem on $\mathscr{E}_1$-precompact sets of right-hand sides in C. It will be shown that if D is $\mathscr{E}_1$-precompact in C and $\mathscr{E}_1$ generates a topology of continuous dependence on it, then any topology on C which generates a topology of continuous dependence on D gives a stronger relative topology (on D) than the relativisation of $\mathscr{E}_1$ to D. This claim is a direct consequence of Proposition 3.1, which is established below. Notice that one way of ensuring that $\mathscr{E}_1$ generates a topology of continuous dependence on D is to have D satisfy the uniform integrability condition in Proposition 2.1.

Proposition 3.1. *Let D be a set of right-hand sides which is $\mathscr{E}_1$-precompact in C, and on which $\mathscr{E}_1$ generates a topology of continuous dependence. If, for each pair $(\{f_\alpha\}, f)$, where $\{f_\alpha\}$ is a net in D, and f is in C, the following property $(*)$ is true, then*

$$\mathscr{E}_1 - \lim_\alpha f_\alpha = f.$$

$(*)$ *For each initial data point (t_0, x_0) in $T \times \mathbb{R}^n$, and subnet $\{f_\beta\}$ of $\{f_\alpha\}$, let $\{x(t_0, x_0; f_\beta)(\cdot)\}$ be a net of maximally defined solutions of the differential equations*

$$\dot{x} = f_\beta(t, x) \qquad \textit{subject to } x(t_0) = x_0$$

for each β. Then there exists a maximally defined solution $x(t_0, x_0; f)(\cdot)$ of the equation

$$\dot{x} = f(t, x) \qquad \textit{subject to } x(t_0) = x_0$$

and a subnet of $\{f_\gamma\}$ of $\{f_\beta\}$ such that

$$\lim_\gamma x(t_0, x_0; f_\gamma)(\cdot) = x(t_0, x_0; f)(\cdot)$$

uniformly on compact subsets of the maximal domain of definition of $x(t_0, x_0; f)(\cdot)$.

Proof. Let $\{f_\alpha\}$ be an arbitrary net contained in D and let f be a function in C such that property $(*)$ holds for the pair $(\{f_\alpha\}, f)$. To establish the proposition, it is sufficient to prove that each subnet of $\{f_\alpha\}$ contains a further subnet which is $\mathscr{E}_1$ convergent to the limit f. Thus fix an arbitrary subnet $\{f_\beta\}$ of $\{f_\alpha\}$. Since D is $\mathscr{E}_1$-compact, there exists some further subnet

$\{f_\gamma\}$ of $\{f_\beta\}$ and some f_1 in C such that $\mathscr{T}_1 - \lim_\gamma f_\gamma = f_1$. Suppose that $f \neq f_1$. The rest of the proof is concerned with showing that this supposition leads to a contradiction. It is given in a series of paragraphs.

(1) Fix an initial data point (t_0, x_0) in $T \times \mathbb{R}^n$, and let $\{(x(t_0, x_0; f_\gamma)(\cdot)\}$ be a net of maximally defined solutions of the differential equations

$$\dot{x}(t) = f_\gamma(t, x(t)) \qquad \text{subject to } x(t_0) = x_0 \tag{3.1}$$

for each γ. Since $\mathscr{T}_1$ generates a topology of continuous dependence on D, there exists a subnet $\{x(t_0, x_0; f_\delta)(\cdot)\}$ of $\{x(t_0, x_0; f_\gamma)(\cdot)\}$ and a maximally defined solution $x(t_0, x_0; f_1)(\cdot)$ of the differential equation

$$\dot{x}(t) = f_1(t, x(t)) \qquad \text{subject to } x(t_0) = x_0 \tag{3.2}$$

such that

$$\lim_\delta x(t_0, x_0; f_\delta)(\cdot) = x(t_0, x_0; f_1)(\cdot), \tag{3.3}$$

where the convergence is uniform on compact subintervals of the maximal domain of definition of $x(t_0, x_0; f_1)(\cdot)$. Now in view of the fact that condition $(*)$ holds for the pair $(\{f_\alpha\}, f)$ and therefore also for the pair $(\{f_\delta\}, f)$, there exists a subnet $\{x(t_0, x_0; f_\varepsilon)(\cdot)\}$ of $\{x(t_0, x_0; f_\delta(\cdot)\}$ and some maximally defined solution $x(t_0, x_0; f)(\cdot)$ of the differential equation

$$\dot{x}(t) = f(t, x) \qquad \text{subject to } x(t_0) = x_0 \tag{3.4}$$

such that

$$\lim_\varepsilon x(t_0, x_0; f_\varepsilon)(\cdot) \equiv x(t_0, x_0; f)(\cdot) \tag{3.5}$$

the convergence being uniform on compact subintervals of the maximal domain of definition of $x(t_0, x_0; f)(\cdot)$. Thus,

$$x(t_0, x_0; f)(\cdot) = x(t_0, x_0; f_1)(\cdot). \tag{3.6}$$

Now,

$$x(t_0, x_0; f)(t) = x_0 + \int_{t_0}^t f(\tau, x(t_0, x_0; f)(\tau)\, d\tau$$

for all t in the maximal domain of definition of $x(t_0, x_0; f)(\cdot)$. Setting

$$\delta f(t, x) = f_1(t, x) - f(t, x)$$

gives

$$x(t_0, x_0; f_1)(t) = x_0 + \int_{t_0}^{t} f_1(\tau, x(t_0, x_0; f_1)(\tau))\, d\tau$$
$$= x_0 + \int_{t_0}^{t} f(\tau, x(t_0, x_0; f)(\tau))\, d\tau$$
$$+ \int_{t_0}^{t} \delta f(\tau, x(t_0, x_0; f)(\tau))\, d\tau$$

whence the following claim is true: for each initial data point (t_0, x_0) in $T \times \mathbb{R}^n$, there exists a maximally defined solution $x(t_0, x_0; f)(\cdot)$ of (3.4) such that

$$\int_{t_0}^{t} \delta f(\tau, x(t_0, x_0; f)(\tau))\, d\tau = 0 \tag{3.7}$$

for all t in the maximal domain of definition of $x(t_0, x_0; f)(\cdot)$.

(2') Since $f_1 \neq f$, there exists some $\bar{x}_0$ in $\mathbb{R}^n$ and a set E in Σ of positive measure, such that $\delta f(t, \bar{x}_0) \neq 0$ for all t in E (this follows from the definition of the equivalence classes in C). Now fix $h > 0$ such that if $|t' - t| < h$ and t' and t are in T, then

$$\left| \int_{t}^{t'} f(\tau, x(t))\, d\tau \right| < \tfrac{1}{2}$$

for all x in X which satisfy $|\bar{x}_0 - x(t)| < 1$, for all t in T. (In view of condition 3 in the definition of C, this choice of h is always possible.) Now fix any t_0 in T and let $x(t_0, \bar{x}_0; f)(\cdot)$ be a maximal solution for (3.4) at $x_0 = \bar{x}_0$, such that (3.7) is true. Then it follows that

$$|x(t_0, \bar{x}_0; f)(t) - \bar{x}_0| < 1$$

for all t in $[t_0 - h, t_0 + h] \cap T$. (If this is not the case then with no loss in generality it may be assumed that there exists some s in $[t_0, t_0 + h] \cap T$ such that

$$|x(t_0, \bar{x}_0; f)(s) - \bar{x}_0| = 1.$$

Let

$$s' = \inf\{s \text{ in } [t_0, t_0 + h];\ |x(t_0, \bar{x}_0; f)(s) - \bar{x}_0| = 1\}.$$

Then, by definition of h, it follows that

$$|x(t_0, \bar{x}_0; f)(t) - \bar{x}_0| = \left| \int_{t_0}^{t} f(\tau, x(t_0, \bar{x}_0; f)(\tau))\, d\tau \right| < \tfrac{1}{2}$$

for all t in $[t_0, s')$, which contradicts the definition of s'.) Therefore, for each t_0 in T, the maximal domain of definition of $x(t_0, \bar{x}_0; f)(\cdot)$ includes the interval $[t_0 - h, t_0 + h] \cap T$, and, by (3.7),

$$\int_{t_0}^{t} \delta f(\tau, x(t_0, \bar{x}_0; f)(\tau))\, d\tau = 0$$

for all t in $[t_0 - h, t_0 + h] \cap T$. Thus, the following claim is valid; there exists an $h > 0$ such that for each t_0 in T, a maximal solution $x(t_0, \bar{x}_0; f)(\cdot)$ exists for (3.4) (at $x_0 = \bar{x}_0$) for which it is true that

$$\delta f(t, x(t_0, \bar{x}_0; f)(t)) = 0 \tag{3.8}$$

a.e. on $[t_0 - h, t_0 + h] \cap T$.

(3) In view of the choice of $\bar{x}_0$ made in the previous paragraph, there is no loss in generality in supposing that for some i in $\{1 \cdots n\}$ it is true that $\delta f^i(t, \bar{x}_0) > 0$ for all t in E, where $\mu(E) > 0$ (here δf^i denotes the ith component of δf). Let $\tilde{\varepsilon} \in (0, \min(h, \mu(E)))$. Then by the Scorza–Dragoni theorem [9, Chap. VIII] there exists an open set $E_{\tilde{\varepsilon}}$ in T such that $\mu(E_{\tilde{\varepsilon}}) < \tilde{\varepsilon}$ and the restriction of δf^i to $(T \sim E_{\tilde{\varepsilon}}) \times \mathbb{R}^n$ is continuous. Since $(E \sim E_{\tilde{\varepsilon}}) \subset (T \sim E_{\tilde{\varepsilon}})$ and $\mu(E \sim E_{\tilde{\varepsilon}}) > 0$, it follows that a $\bar{t}_0$ can be chosen in $(E \sim E_{\tilde{\varepsilon}})$ to be a Lebesgue density point of $(T \sim E_{\tilde{\varepsilon}})$. Thus

$$\lim_{\eta \downarrow 0} \frac{\mu((T \sim E_{\tilde{\varepsilon}}) \cap [t_0 - \eta, t_0 + \eta])}{2\eta} = 1$$

whence $\mu((T \sim E_{\tilde{\varepsilon}}) \cap [t_0 - \eta, t_0 + \eta]) > 0$ for all $\eta > 0$. Let $\delta f^i(\bar{t}_0, \bar{x}_0) = a > 0$. Then there exists some $\eta_i > 0$ such that if $|t - \bar{t}_0| < \eta_i$ where t is in $(T \sim E_{\tilde{\varepsilon}})$ and $|x - \bar{x}_0| < \eta_i$ then $\delta f^i(t, x) > a/2$. Now choose some η_2 in $(0, h)$ such that $|\bar{x}_0 - x(t_0, \bar{x}_0; f)(t)| < \eta_1$ for all t in $[\bar{t}_0 - \eta_2, \bar{t}_0 + \eta_2] \cap T$. Let $\eta_3 = \min(\eta_1, \eta_2)$. Then clearly $\delta f^i(t, x(\bar{t}_0, \bar{x}_0; f)(t)) > a/2$ for all t in $[\bar{t}_0 - \eta_3, \bar{t}_0 + \eta_3] \cap (T \sim E_{\tilde{\varepsilon}})$. But this contradicts the fact that $\delta f(t, x(t_0, \bar{x}_0; f)(t)) = 0$ a.e. on $[t_0 - h, t_0 + h] \cap T$ (see (3.8)). Therefore, the supposition that $f_1 \neq f$ is false and the proposition follows. ∎

COROLLARY 3.1. *If D is a $\mathscr{E}_1$-precompact subset of C and $\mathscr{E}_1$ generates a topology of continuous dependence on D, then so also does $\mathscr{E}$. Moreover, any topology on C generating a topology of continuous dependence on D necessarily gives a stronger relativisation to D than does $\mathscr{E}$.*

Proof. Since $\mathscr{E}$ is weaker than $\mathscr{E}_1$ and $\mathscr{E}$ is also a Hausdorff topology, the following observation is clear: if $\{f_\alpha\}$ is a net in D and f is a point in C such that $(\{f_\alpha\}, f)$ belongs to the Moore–Smith convergence class generated by $\mathscr{E}$ on C, then it is also in the Moore–Smith convergence class of $\mathscr{E}_1$ on C, i.e., $\mathscr{E}$ and $\mathscr{E}_1$ generate the same relative topology on D, whence $\mathscr{E}$ generates a topology of continuous dependence on D when $\mathscr{E}_1$ does so. The second claim of the corollary is a direct consequence of Proposition 3.1.

The compactness of D plays an essential role in the proof of Proposition 3.1. Indeed, when the set of right-hand sides is not $\mathscr{E}_1$-precompact in C, then it is possible to have convergence of the solutions of a sequence of differential equations without the corresponding sequence of right-hand sides converging in the topology $\mathscr{E}_1$ to the right-hand side for the limiting equation. Situations of this kind are not covered by Propositions 2.1 and 3.1, and the problem of characterizing a weakest topology of continuous dependence (if any) under these conditions will require methods different from those used above. This problem has apparently not been solved. An example of such a collection of right-hand sides occurs in [10] in connection with the theory of generalised or Kurzweil ordinary differential equations. In this example the sequence of equations is given by

$$\dot{x} = k^{1-\alpha} \cdot x \cdot \sin(k \cdot t) + k^{1-\beta} \cdot \cos(k \cdot t), \qquad k = 1, 2...$$

where the right-hand sides are defined on $[0, 1] \times \mathbb{R}^n$, and the parameters α and β are fixed at some values in the range $(0, 1)$ such that $\alpha + \beta > 1$.

The limiting equation is given by the right-hand side

$$f_\infty(t, x) = 0 \qquad \text{for all } (t, x) \text{ in } [0, 1].$$

Each of these equations has a unique solution which can be written in the following form for the initial data point (t_0, x_0):

$$\begin{aligned} x_k(t_0, x_0)(t) = {} & \exp(k^{-\alpha} \cdot \sin(k \cdot t)) \cdot (x_0 \cdot \exp\{-k^{-\alpha} \cdot \sin(k \cdot t_0)\}) \\ & + k^{-\beta}(\cos(k \cdot t_0) - \cos(k \cdot t)) \\ & - k^{1-\alpha-\beta} \cdot \int_{t_0}^{t} \sin^2(k, \tau)\, d\tau + O(k^{1-2\alpha-\beta}). \end{aligned}$$

Clearly, the sequence $\{x_k(t_0, x_0)(\cdot)\}$ converges uniformly to the function which is identically equal to x_0 on the unit interval, and which is the solution of the limiting equation. However, it is not true that $\{f_k\}$ converges to f_∞ in the $\mathscr{E}_1$ topology, or even that $\{f_k\}$ is $\mathscr{E}_1$-precompact in C. To see this, consider the sequence of continuous functions $\{k^{\alpha-1} \cdot \sin(k \cdot t)\}$ which are defined on the unit interval. For each t' and t'' in $[0, 1]$, it is clear that

$$\int_{t'}^{t''} f_k(t, k^{\alpha-1} \cdot \sin(k \cdot t))\, dt \to \frac{(t'' - t')}{2}$$

as $k \to \infty$. Since the functions $\{k^{1-\alpha} \cdot \sin(k \cdot t)\}$ converge to zero uniformly on the unit interval, the non-convergence of $\{f_k\}$ to f_∞ in the $\mathscr{E}_1$ topology is established. To see that $\{f_k\}$ is not $\mathscr{E}_1$-precompact in C, it is sufficient to note that it is $\mathscr{E}$-convergent to f_∞. Thus $\mathscr{E}$ and $\mathscr{E}_1$ disagree on $\{f_k\}$; i.e., $\{f_k\}$ is not precompact in the $\mathscr{E}_1$-topology.

4. Compact Sets of Right-Hand Sides

In order to use Proposition 3.1 it is necessary to determine when a given set of right-hand sides is $\mathscr{E}_1$-precompact in C. In the present section conditions will be given which ensure that the compactness holds. The conditions will then be used to study an example of a collection of right-hand sides on which $\mathscr{E}$ generates a weakest topology of continuous dependence.

The conditions which will be shown to imply the compactness of a set $D \subset C$ are as follows:

D1: for each $\eta > 0$ the set of functions

$$\{f(\cdot x(\cdot)); \|x\| \leqslant \eta, f \text{ in } D\}$$

is uniformly integrable;

D2: for each compact set K in X and each set E in Σ, the set of functions

$$\left\{x \to \int_E f(t, x(t))\, dt : f \text{ in } D\right\}$$

is equicontinuous on K;

D3: for each $\varepsilon > 0$ and net $\{f_\alpha\}$ in D having the property that $\{f_\alpha(\cdot, x(\cdot))\}$ is L_1-weakly convergent for each x in X, there exists a set G_ε in Σ such that $\mu(G_\varepsilon) < \varepsilon$ and the set of functions

$$\left\{x \to \frac{1}{\mu(E)} \lim_\alpha \int_E f_\alpha(t, x(t))\, dt; E \text{ in } \Sigma, \mu(E) > 0, E \cap G_\varepsilon = \varnothing\right\}$$

is equicontinuous in each compact set K in X.

Proposition 4.1. *If a set $D \subset C$ satisfies conditions* D1, D2, *and* D3 *then it is $\mathscr{E}_1$-precompact in C.*

Proof. The claim will be established by showing that an arbitrary net in D contains a subnet which converges uniformly on compact subsets of X to a limit which belongs to C.

(1) Let $\{f_\alpha\}$ be a net contained in D and let $A = \{x_j\}$ be a dense sequence in X. Since $\{f_\alpha(\cdot, x_j(\cdot))\}$ is uniformly integrable for each j (see D1),

there exists a sequence of subnets $S_j = \{f_\alpha(\cdot, x_j(\cdot)); \alpha \text{ in } \mathscr{N}_j\}$ such that S_{j+1} is a subnet of S_j and S_j converges in the weak topology of L_1 to an L_1-limit which is denoted by $g(\cdot, x_j)$ for each j. (The above claim follows from the Eberlein–Smulian theorem [11] and the fact that a uniformly integrable set of L_1-functions is weakly countably compact in L_1.) By a standard argument (see [8, Theorem 2.4]) construct a "diagonal net" $\{f_\beta\}$ such that $\{f_\beta(\cdot, x_j(\cdot))\}$ converges weakly to $g(\cdot, x_j))$ for each j. The diagonal net is a subnet of $\{f_\alpha\}$ and the remaining sections of the proof are devoted to showing that it converges uniformly on compact subsets of X to a limit in C.

(2) Define the function $\phi: A \times \Sigma \to \mathbb{R}^n$ as follows:

$$\phi(x_j, E) = \lim_\beta \int_E f_\beta(t, x_j(t))\, d. \tag{4.1}$$

For each E, there is a unique continuous extension of $\phi(\cdot, E)$ from A to X. This is clear when $\mu(E) = 0$; for the case when E is of positive measure, fix any x in X, and let $\{x_r\}$ be a sequence in A which converges to x. The condition D2 ensures that $\{\phi(x_r, E)\}$ is a Cauchy sequence in $\mathbb{R}^n$, and it is easily verified that if one defines $\phi(x, E) = \lim_r \phi(x_r, E)$ then the function thus obtained is the unique continuous extension of $\phi(\cdot, E)$ from A to X. Moreover, since

$$\lim_r \int_E f_\beta(t, x_r(t))\, dt = \int_E f_\beta(t, x(t))\, dt$$

uniformly with respect to β (see D2), an interchange of limits is permissible, and therefore

$$\begin{aligned}\phi(x, E) &= \lim_r \lim_\beta \int_E f_\beta(t, x_r(t))\, dt \\ &= \lim_\beta \lim_r \int_E f_\beta(t, x_r(t))\, dt \\ &= \lim_\beta \int_E f_\beta(t, x(t))\, dt \end{aligned} \tag{4.2}$$

for each E in Σ and x in X.

Now it will be shown that each $\phi(x, \cdot)$ is a countably additive measure which admits a Radon–Nikodym derivative $g(\cdot, x)$ (with respect to $\mu(\cdot)$) with the property that for almost all t in T, $g(t, \cdot)$ is continuous on X. Then $g(\cdot, \cdot)$ will be used to obtain a function $f(\cdot, \cdot)$ in C which will be shown to be the limit (in the sense of the topology $\mathscr{E}_1$) of the subnet $\{f_\beta\}$. This will establish the proposition

(3) Since $\phi(x_j, E) = \int_E g(t, x_j)\, dt$ (4.3)

where $g(\cdot, x_j)$ is an L_1-function for each j, it follows that $\phi(x_j, \cdot)$ is countably additive on Σ. Fix an x and let $\{x_r\}$ be a sequence in A converging to x. Now $\lim_r \phi(x_r, E) = \phi(x, E)$ for each E in Σ, and because $g(\cdot, x_r)$ is in L_1, $\phi(x_r, \cdot)$ is of bounded variation on Σ. Thus the Vitali–Hahn–Saks theorem [11] shows that the set function $\phi(x, \cdot)$ is countably additive on Σ. Moreover, $\phi(x, \cdot)$ is clearly absolutely continuous with respect to $\mu(\cdot)$, and since $\{x_r\}$ is bounded in X condition D1 implies that $\phi(x, \cdot)$ is a finite measure on Σ. Therefore, for each x, $\phi(x, \cdot)$ is differentiable with respect to $\mu(\cdot)$ a.e. on T. Let $T(x_j)$ denote the full-measure subset of T such that the limit

$$\lim_m \frac{\phi(x_j, [t, t + 1/m))}{(1/m)} \tag{4.4}$$

is well defined for each t in $T(x_j)$. The existence of $T(x_j)$ is given by the theorem on the differentiation of measures (e.g., [12, Theorem 8.6]), which also claims that the limit in (4.4) coincides a.e. with $g(\cdot, x_j)$ on T. Set $T' = \bigcap_{j=1}^{\infty} T(x_j)$. Let $\{\varepsilon_l\}$ be a sequence of real numbers decreasing monotonically to zero. In view of condition D3, there exists for each l a set G_l of μ-measure less than ε_l such that the collection of functions

$$\left\{ x \to \frac{\phi(x, E)}{\mu(E)}; \mu(E) > 0, E \cap G_l = \varnothing \right\}$$

is equicontinuous in each compact subset of X. With no loss in generality, G_l may be taken to be open, and assumed to be the union of the intervals (a_i, b_i), $i = 1, 2...$, where $a_1 < b_1 \leqslant a_2 < b_2 \leqslant a_3 \cdots$. Let Q_l be the union of the semi-closed intervals $[b_i, a_{i+1})$ for all i such that $b_i < a_{i+1}$. Again fix an x and let $\{x_r\}$ be a sequence in A which converges to x. Fix a t in $T' \cap Q_l$. Then, in view of the equicontinuity noted above,

$$\lim_r \frac{\phi(x_r, [t, t + 1/m) \cap Q_l)}{\mu([t, t + 1/m) \cap Q_l)} = \frac{\phi(x, [t, t + 1/m) \cap Q_l)}{\mu([t, t + 1/m) \cap Q_l)} \tag{4.5}$$

uniformly with respect to m, and from the definition of T',

$$\lim_m \frac{\phi(x_r, [t, t + 1/m) \cap Q_l)}{\mu([t, t + 1/m) \cap Q_l)} = g(t, x_r) \tag{4.6}$$

for each r. Thus the iterated limits $\lim_r \lim_m$ and $\lim_m \lim_r$ exist and are equal, whence

$$\begin{aligned}\lim_r g(t, x_r) &= \lim_r \lim_m \frac{\phi(x_r, [t, t+1/m) \cap Q_l)}{\mu([t, t+1/m) \cap Q_l)} \\ &= \lim_m \lim_r \frac{\phi(x_r, [t, t+1/m) \cap Q_l)}{\mu([t, t+1/m) \cap Q_l)} \\ &= \lim_m \frac{\phi(x, [t, t+1/m))}{(1/m)}. \end{aligned} \tag{4.7}$$

Defining

$$\begin{aligned} g_l(t, x) &= \lim_m \frac{\phi(x, [t, t+1/m))}{(1/m)} \qquad \text{for } t \text{ in } T' \cap Q_l \\ &= 0 \qquad \text{for } t \text{ in } (T \sim T') \cap Q_l \end{aligned}$$

gives a function on $Q_l \times X$ to $\mathbb{R}^n$ such that $g_l(\cdot, x)$ is measurable on Q_l for each x, and $g_l(t, \cdot)$ is continuous on X for each t in Q_l (in fact it is clearly the continuous extension of $g(t, \cdot)$ from A to X). Moreover, $g_l(t, x)$ coincides a.e. with the Radon–Nikodym derivative of $\phi(x, \cdot)$ with respect to $\mu(\cdot)$ so that for each x in X,

$$\phi(x, E \cap Q_l) = \int_{E \cap Q_l} g_l(t, x)\, dt \tag{4.8}$$

for all E in Σ.

A standard argument (see [13, Theorem III.2.5], for example) may now be used to extend the functions $g_l(\cdot, x)$ from Q_l to T as follows. Define a collection of subsets $\{R_l\}$ of T by $R_1 = Q_1$ and $R_l = Q_l \sim \bigcup_{k=1}^{l-1} Q_k$, and define the function $g(\cdot, \cdot)$ from $T \times X$ into $\mathbb{R}^n$ by

$$\begin{aligned} g(t, x) &= g_l(t, x) \qquad \text{for } t \text{ in } T' \cap R_l \text{ and } x \text{ in } X \\ g(t, x) &= 0 \qquad \text{for } t \text{ in } (T \sim T') \cup \left[\bigcap_{l=1}^{\infty} (T \sim R_l) \right] \\ &\qquad \text{and } x \text{ in } X. \end{aligned}$$

The monotone convergence theorem and the fact that $\phi(x, \cdot)$ is of bounded variation implies that $g(\cdot, x)$ is an L_1-function for each x, and from the dominated convergence theorem it follows that

$$\phi(x, E) = \int_E g(t, x)\, dt \tag{4.9}$$

for each x in X and E in Σ. Since T' and $\bigcup_{l=1}^{\infty} R_l$ are of full measure in T, $g(t, \cdot)$ is clearly continuous on X for almost all t. In view of (4.2),

$$\lim_\beta \int_E f_\beta(t, x(t))\, dt = \int_E g(t, x)\, dt \tag{4.10}$$

for each E in Σ and x in X, and condition D2 and the fact that pointwise convergence of an equicontinuous set of functions implies uniform convergence, shows that the convergence in (4.10) is uniform on compact subsets of X. To finish the proof it remains to show that $g(t, x)$ actually corresponds to an element of C. This will be done by showing that there exists some $f(\cdot, \cdot)$ in C such that

$$f(t, x(t)) = g(t, x) \qquad \text{a.e. in } T,$$

for each x in X.

(4) Fix an x in X and a t in $T' \cap [\bigcup_{l=1}^{\infty} R_l]$, and let $K = \{x(\cdot)\} \cup \{x(t)(\cdot);\ t \text{ in } T\}$, where, for each t, $x(t)(\cdot)$ denotes the function in X which is constant at the value $x(t)$. Since K is a compact set in X, condition D3 implies that the set of functions

$$\left\{ x \to \frac{\phi(x, [t, t+1/m))}{(1/m)};\ m \right\}$$

is equicontinuous on K. Since $\phi(x, [t, t+1/m))$ does not depend on the restriction of $x(\cdot)$ to the complement of $[t, t+1/m)$, for each $\varepsilon < 0$ there exists a $\delta(\varepsilon) > 0$ such that if $|x(\tau) - x(t)(\cdot)| < \delta(\varepsilon)$ for all τ in $[t, t+1/m)$ then

$$\left| \frac{\phi(x, [t, t+1/m)) - \phi(x(t)(\cdot), [t, t+1/m))}{(1/m)} \right| < \varepsilon$$

for each m, and thus there exists an $m(\varepsilon)$ such that $m > m(\varepsilon)$ implies that

$$\left| \frac{\phi(x, [t, t+1/m)) - \phi(x(t)(\cdot), [t, t+1/m))}{(1/m)} \right| < \varepsilon.$$

Moreover, t is in $T' \cap R_l$ for some l, whence

$$\lim_m \frac{\phi(x, [t, t+1/m))}{1/m} = g_l(t, x) = g(t, x)$$

and since ε is arbitrary,

$$\lim_m \frac{\phi(x(t)(\cdot), [t, t+1/m))}{1/m} = g(t, x).$$

Therefore, by the definition of $g(t, x(t)(\cdot))$, it follows that

$$g(t, x) = g_l(t, x(t)(\cdot)) = g(t, x(t)(\cdot)).$$

Now define the function $f: T \times \mathbb{R}^n \to \mathbb{R}^n$ as follows: for each x in $\mathbb{R}^n$,

$$f(t,x) = g(t, \hat{x}(\cdot)) \qquad \text{for } t \text{ in } T' \left. \bigcup_{l=1}^{\infty} R_l \right]$$
$$= 0 \qquad \text{otherwise.}$$

(Here, $\hat{x}(\cdot)$ denotes the function in X which is constant at the value x.) Then $f(t, x(t)) = g(t, x)$ a.e. on T, and moreover,

$$\phi(x, E) = \int_E f(t, x(t))\, dt$$

for all E in Σ. In view of (4.10), it follows that $f(\cdot, \cdot)$ is the $\mathscr{E}_1$-limit of the subnet $\{f_\beta\}$. Moreover, it is clear that $f(t,x)$ is measurable on T and continuous on $\mathbb{R}^n$, and D1 ensures that $\{f(\cdot, x(\cdot)); \|x(\cdot)\| \leqslant \eta\}$ is uniformly integrable for $\eta > 0$. Thus f is in C and the claim follows. ∎

EXAMPLE. *Let* $\{z_\alpha(\cdot)\}$ and $\{e_\alpha(\cdot)\}$, $1 \leqslant \alpha \leqslant 2$, be sets of L_∞-functions defined on $[0, 1]$ such that $\|z_\alpha(\cdot)\|_\infty \leqslant M$ and $\frac{1}{2} \leqslant e_\alpha(t) \leqslant 1$ a.e. on T for each α. Consider the set of right-hand sides $\{f_\alpha\}$ defined as follows:

$$f_\alpha(t,x) = |x - z_\alpha(t)|^{e_\alpha(t)}$$

for $1 \leqslant \alpha \leqslant 2$, where x is real. Define

$$v(t, \alpha, x, z) = |x - z|^{e_\alpha(t)}$$

for t in the unit interval, $1 \leqslant \alpha \leqslant 2$, and x and z real. To show that $\{f_\alpha\}$ satisfies conditions D1 to D3, the following simple assertions are needed:

Claim 1. The set of functions

$$\{x \to v(t, \alpha, x, 0);\ 1 \leqslant \alpha \leqslant 2, t \leqslant 1\}$$

is equicontinuous on $\mathbb{R}$.

Proof. The equicontinuity at $x = 0$ is immediate. For $x > 0$, equicontinuity at x follows from the fact that when $x' > 0$, $|v(t, \alpha, x', 0) - v(t, \alpha, x, 0)| \leqslant |x - x'| \cdot (e_\alpha(t)) \cdot (\max |u|^{e_\alpha(t)-1}; x \leqslant y \leqslant x')$ and the coefficient of $|x - x'|$ on the right of the inequality is majorised by a finite value when t varies over the unit interval, α varies on $[1, 2]$ and x' is confined to a sufficiently small interval centered on x. The claim follows.

Claim 2. For $c > 0$ the set of functions

$$\{x \to v(t, \alpha, x, z):\ 1 \leqslant \alpha \leqslant 2, t \leqslant 1, |z| \leqslant c\}$$

is equicontinuous.

Proof. In view of Claim 1, the set of functions

$$\{x \to v(t, \alpha, x, 0);\ 1 \leqslant \alpha \leqslant 2, t \geqslant 1\}$$

is uniformly equicontinuous on $[-2c, 2c]$. From the form of $v(\cdot)$, the set of functions

$$\{x \to v(t, \alpha, x, z);\ 1 \leqslant \alpha \leqslant 2 \cdot t \leqslant 1 \cdot |z| \leqslant c\}$$

is also uniformly equicontinuous on $[-c, c]$. The claim follows.

Considering the set of right-hand sides, Claim 2 implies that for $\varepsilon > 0$ and $x(\cdot)$ in $C[0, 1]$, the space of continuous functions on the unit interval, there exists some $\delta(x(\cdot), \varepsilon)$ such that $\|x(\cdot) - x'(\cdot)\| < \delta(x(\cdot), \varepsilon)$ implies that

$$|f_\alpha(t, x(t)) - f_\alpha(t, x'(t))| < \varepsilon$$

for all t in $[0, 1]$ and $1 \leqslant \alpha \leqslant 2$. Therefore, conditions D2 and D3 are satisfied. Moreover D1 is obviously satisfied because for each $\eta > 0$, $|f_\alpha(t, x(t))|$ is majorised by some finite value as t and α vary on $[0, 1]$ and $[1, 2]$, respectively, and $x(\cdot)$ varies on the ball of radius η in $C[0, 1]$ centered on the origin. Therefore, D is $\mathscr{F}_1$-precompact in C (Proposition 4.1) whence $\mathscr{F}$ and $\mathscr{F}_1$ generate identical relative topologies on it, $\mathscr{F}$ gives a topology of continuous dependence on D (Prop. 2.1), and is the weakest such topology of continuous dependence that can be defined on D (Proposition 3.1).

5. General Remarks

This note considers the problem of formulating a weakest topology on a collection of differential equations, such that continuous dependence is obtained globally with respect to the initial data. Results of this kind were first obtained in [6]. It is possible to study the continuous dependence problem from an alternative point of view, which emphasises continuous dependence jointly with respect to the right-hand side and the initial data. For example, the results in [2] are of this kind, and pertain to Volterra integral equations as well as ordinary differential equations. A very interesting aspect of the continuous dependence problem in this form is that in general there fails to exist a weakest topology on the given set of right-hand sides such that the function taking the right-hand side and "initial trajectory" into the set of solution trajectories is continuous. An example of this situation is given in [2], where a set of differential equations illustrating this phenomenon is constructed in terms of the Schauder basis for $C[0, 1]$. The results in [2] are extended to an abstract theory in [14], which covers the case of general operator equations, and it is also shown that although there generally fails to exist a weakest topology giving continuous dependence jointly with respect to the right-hand side and initial trajectory, there nevertheless exists, under general conditions, a richest convergence structure on the given set of operators, with respect to which the desired joint continuous dependence holds. (A convergence structure is distinguished from a topology by the fact that it does not necessarily have to satisfy the iterated limit condition. See [8, Chap. 2] and [14, p. 147].)

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